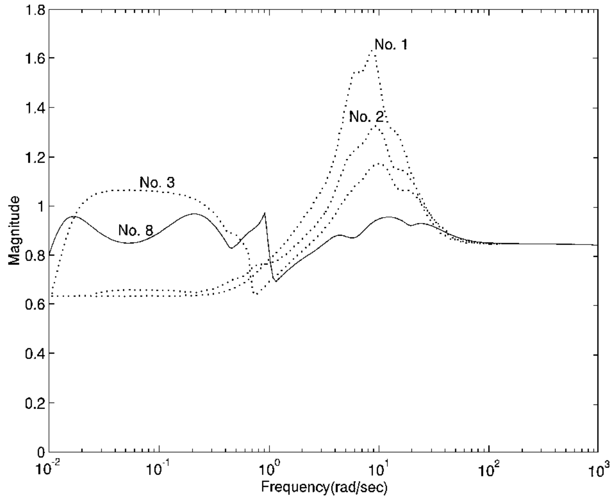

Fig. 1 Upper bounds of $\mu_{\hat{\Delta}}[F_l(P, K_i)(j\omega)]$ for iterations 1, 2, 3, and 8.

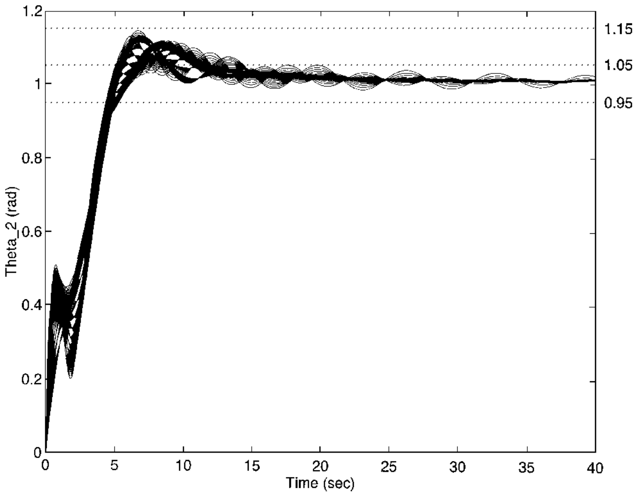

Fig. 2 Step responses of 121 perturbed satellite plants with mixed μ controller.

model reduction methods, it is possible to reduce the controller order to a lower level with acceptable degradation in control performance. However, using controller approximation with fewer states implies an increase in μ. Still, there are open numerical problems in connection with actual implementation in μ-synthesis.

2) The robust performance (RP) condition in Eq. (8) is identical to a robust stability (RS) condition with an additional perturbation block Δ_P. In other words, the RP condition is more severe than the RS condition, and so there is no doubt of RS in this illustrative satellite case, where the RP condition is satisfied.

V. Conclusions

We extend the H_2-based loop-shaping method to mixed μ-synthesis. A noncollocated satellite's attitude control design is formulated to the mixed μ-synthesis problem. The resulting controller with RP property from the μ criterion is obtained by a sequence of weighted H_2 optimizations. Also, the simulation results show that the time-domain performance specifications on settling time and overshoot are completely satisfied under real parameter variations. This successful design stresses the superior ability of the H_2-based loop-shaping method for mixed μ-synthesis.

References

[1]Braatz, R. P., Young, P. M., Doyle, J. C., and Morari, M., "Computational Complexity of μ Calculation," *IEEE Transactions on Automatic Control*, Vol. 39, No. 5, 1994, pp. 1000–1002.

[2]Young, P. M., "Controller Design with Mixed Uncertainties," *Proceedings of the American Control Conference* (Baltimore, MD), 1994, pp. 2333–2337.

[3]Steen, T. C., "Mixed μ Synthesis—μ-K Iteration," *System Identification and Robust Control—A Case Study Approach*, Springer–Verlag, London, 1996, pp. 81–86.

[4]Yang, C.-D., and Chang, C.-Y., "Synthesis of Optimal H_∞ Controllers via H_2-Based Loop-Shaping Design," *Journal of Guidance, Control, and Dynamics*, Vol. 19, No. 4, 1996, pp. 878–885.

[5]Yang, C.-D., and Chang, C.-Y., "μ-Synthesis Using Linear Quadratic Gaussian Controllers," *Journal of Guidance, Control, and Dynamics*, Vol. 19, No. 4, 1996, pp. 886–892.

[6]Franklin, G. F., Powell, J. D., and Emami-Naeini, A., "Design of a Satellite's Attitude Control," *Feedback Control of Dynamic Systems*, 3rd ed., Addison–Wesley, Reading, MA, 1994, pp. 669–684.

Linearization of a Six-Degree-of-Freedom Missile for Autopilot Analysis

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Introduction

TYPICALLY, the design of autopilots for missiles is based on a loop-at-a-time philosophy; that is, the lateral channels (yaw and pitch) are designed, and then the roll channel is designed separately.[1] Most of these autopilots have a fixed structure whose gains are then scheduled upon flight conditions such as altitude, Mach, and angle of attack. Gain selection is accomplished by examining both the performance and the relative stability of a linear representation of the missile at various points in the flight envelope. Most often, classical phase and gain margins are used as a measurement of relative stability. The worst-case values of these margins are obtained by breaking a single loop in a coupled lateral-roll model. Such design approaches are insufficient for obtaining autopilots that a priori account for robustness and performance. With the advances in control theory, it is now possible to design multivariable controllers that can account for both relative stability (robustness) and performance.[2] To perform such design and analysis, one must work with a fully coupled dynamic representation of the missile. This Note describes a way to properly linearize the dynamics for a fully coupled, six-degree-of-freedom (DOF), cruciform missile in trim. One accepted definition of trim for missiles that can attain large angles of attack is as follows: 1) the moments acting on the missile are zero and 2) the rate of change of both the angle of attack and the sideslip angle are zero. Note that this definition does not preclude nonzero roll rates at trim. An additional contribution of this Note is the derivation of a technique for determining the initial roll rates that are consistent with the dynamics of a trimmed missile as just defined.

Development of Linear Model

We define the following quantities for a missile with respect to Fig. 1: (x, y, z) are the missile body axes, (u, v, w) the projection of the missile velocity vector onto (x, y, z), and (p, q, r) the projection of the missile angular rate vector onto (x, y, z).

Note that (u, v, w) and (p, q, r) are, in fact, inertial values written with respect to the body axes. The analysis presented here is for a missile that is symmetric about its principal axes; thus, the inertial dyad is $\boldsymbol{I} = \mathrm{diag}([I_{xx} \;\; I_{zz} \;\; I_{zz}])$. We divide the forces and moments that act on the missile into two categories: 1) forces and moments due to missile aerodynamics including canards, wings, or fins; and

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2) forces and moments due to propulsive elements such as rocket motors, reaction jets, or divert thrusters. Let $\boldsymbol{F} = \boldsymbol{F}_{\text{aero}} + \boldsymbol{F}_{\text{prop}}$ be the total force acting on the missile and $\boldsymbol{M} = \boldsymbol{M}_{\text{aero}} + \boldsymbol{M}_{\text{prop}}$ be the total moment acting on the missile. Thus, $\boldsymbol{F} = [F_x \;\; F_y \;\; F_z]^T$ and $\boldsymbol{M} = [l \;\; m \;\; n]^T$, where F_i is the force in the ith direction, l is the rolling moment, m is the pitching moment, and n is the yawing moment. Standard techniques, such as can be found in Ref. 3, are used to develop the following coupled equations of motion for the missile:

$$F_x = \dot{M}u + M(\dot{u} + qw - rv) \tag{1}$$

$$F_y = \dot{M}v + M(\dot{v} + ru - pw) \tag{2}$$

$$F_z = \dot{M}w + M(\dot{w} - qu + pv) \tag{3}$$

$$l = \dot{I}_{xx}p + I_{xx}\dot{p} \tag{4}$$

$$m = \dot{I}_{zz}q + I_{zz}\dot{q} - (I_{zz} - I_{xx})pr \tag{5}$$

$$n = \dot{I}_{zz}r + I_{zz}\dot{r} - (I_{xx} - I_{zz})pq \tag{6}$$

As in Ref. 4, the total angle of attack α_t is defined to be the angle between the velocity vector and the missile centerline; the aerodynamic roll angle ϕ_a is defined to be the angle between the (positive) body z axis and the projection of the velocity vector in the body yz plane. From Fig. 1, one can see that $\alpha_t = \tan^{-1}[\sqrt{(w^2 + v^2)}/u]$ and $\phi_a = \tan^{-1}(v/w)$. Furthermore, the pitch plane angle of attack α is defined to be the angle between the (positive) body x axis and the projection of the velocity vector $\boldsymbol{V}$ in the xz plane. The sideslip angle of attack β is defined to be the angle between the (positive) body x axis and the projection of the velocity vector $\boldsymbol{V}$ in the xy plane. Again, from Fig. 1 one can see that $\alpha = \tan^{-1}(w/u)$ and $\beta = \tan^{-1}(v/u)$. The following relationships exist between the angles α_t, ϕ_a, α, and β:

$$\alpha_t = \tan^{-1}\left[\sqrt{\tan(\alpha)^2 + \tan(\beta)^2}\right], \qquad \phi_a = \tan^{-1}\left[\frac{\tan(\beta)}{\tan(\alpha)}\right]$$

$$\alpha = \tan^{-1}(\tan\alpha_t \cos\phi_a), \qquad \beta = \tan^{-1}(\tan\alpha_t \sin\phi_a)$$

For a missile with a specific orientation (α_t, ϕ_a, Mach, and altitude), trimmed flight occurs when the fin deflections are such that the total moments l, m, and n are zero and the rate of change of the angles of attack α and β is zero. That is, a missile is in trim when l, m, and $n = 0$ and $\dot{\alpha}$ and $\dot{\beta} = 0$. A typical aerodynamic database for a missile will return the aerodynamic forces $F_{x_{\text{aero}}}$, $F_{y_{\text{aero}}}$, and $F_{z_{\text{aero}}}$ and three aerodynamic moments l_{aero}, m_{aero}, and n_{aero} for a given missile orientation and fin deflection. We will assume that mass flow rates due to the propulsive components are constant, and known, at the specific orientation we are analyzing. Thus, for our analysis, $\boldsymbol{F}_{\text{prop}}$ and $\boldsymbol{M}_{\text{prop}}$ are known constants. It is now possible to use the aerodynamic database to determine whether a missile can be trimmed at a specific orientation by iterating on the fin deflections to (hopefully) find a configuration such that the total moments $l = l_{\text{aero}} + l_{\text{prop}}$, $m = m_{\text{aero}} + m_{\text{prop}}$, and $n = n_{\text{aero}} + n_{\text{prop}}$ are zero. Most autopilots are designed by linearizing the six equations of motion for a missile that is trimmed in a specific orientation. This requires that the initial velocity vector $\boldsymbol{V}_0$ and the initial rotational vector $\boldsymbol{\omega}_0$ be determined. Using the formulas for α_t, ϕ_a, α, and β yields

$$\boldsymbol{V}_0 = \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} = \begin{bmatrix} V_t \cos\alpha_t \\ V_t \sin\alpha_t \sin\phi_a \\ V_t \sin\alpha_t \cos\phi_a \end{bmatrix} \tag{7}$$

where $V_t = \sqrt{(u_0^2 + v_0^2 + w_0^2)}$. To determine the initial rotational vector, recall that the initial acceleration vector is known because the total inertial force is the sum of the aerodynamic forces, provided by the aero database, and the propulsive forces, which are assumed to be known constants. Consequently, the initial acceleration vector is calculated simply by dividing the total force by the mass: $\boldsymbol{A}_0 = [a_{x0} \;\; a_{y0} \;\; a_{z0}]^T$, where $a_{i0} = F_{i0}/M_0$. Define the vector $\boldsymbol{T}$ to be a unit vector along the initial velocity vector, $\boldsymbol{T} = \boldsymbol{V}_0/|\boldsymbol{V}_0|$. The projection of the initial acceleration vector $\boldsymbol{A}_0$ along $\boldsymbol{T}$ is given by $\boldsymbol{A}_{T0} = \langle \boldsymbol{A}_0, \boldsymbol{T}\rangle \boldsymbol{T}$. The component of $\boldsymbol{A}_0$ normal to $\boldsymbol{V}_0$ is then given by $\boldsymbol{A}_{N0} = \boldsymbol{A}_0 - \boldsymbol{A}_{T0}$. Define the vector $\boldsymbol{N}$ to be a unit vector normal to the velocity vector, $\boldsymbol{N} = \boldsymbol{A}_{N0}/|\boldsymbol{A}_{N0}|$, so that

$$\boldsymbol{A}_0 = \langle \boldsymbol{A}_0, \boldsymbol{T}\rangle \boldsymbol{T} + |\boldsymbol{A}_{N0}|\boldsymbol{N}$$

Noting that $\boldsymbol{A}_0 = (\dot{M}_0/M_0)\boldsymbol{V}_0 + [\mathrm{d}(\boldsymbol{V}_0)_r/\mathrm{d}t] + (\boldsymbol{\omega}_0 \times \boldsymbol{V}_0)$, where the first two components are tangent to the velocity vector $\boldsymbol{V}_0$ and the third component is normal to the velocity vector yields

$$|\boldsymbol{A}_{N0}|\boldsymbol{N} = (\boldsymbol{\omega}_0 \times \boldsymbol{V}_0)$$

which can be written as

$$\begin{bmatrix} a_{nx} \\ a_{ny} \\ a_{nz} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & w_0 & -v_0 \\ -w_0 & 0 & u_0 \\ v_0 & -u_0 & 0 \end{bmatrix}}_{S(V_0)} \begin{bmatrix} p_0 \\ q_0 \\ r_0 \end{bmatrix} \tag{8}$$

where $S(\boldsymbol{V}_0)$ is a skew symmetric matrix, $S(\boldsymbol{V}_0) = -S(\boldsymbol{V}_0)^T$. The matrix $S(\boldsymbol{V}_0)$ is singular and has a one-dimensional null space, which is given by $\mathfrak{N}[S(\boldsymbol{V}_0)] = ([u_0 \;\; v_0 \;\; w_0]^T)$. In general then, the solution of Eq. (8) for $\boldsymbol{\omega}_0$ can be written as $\boldsymbol{\omega}_0 = \boldsymbol{\omega}_{0r} + \boldsymbol{\omega}_{0n}$, where the vector $\boldsymbol{\omega}_{0r}$ is contained in the range space of $S(\boldsymbol{V}_0)$ and $\boldsymbol{\omega}_{0n}$ is contained in the null space of $S(\boldsymbol{V}_0)$. In terms of the dynamics of the problem, $\boldsymbol{\omega}_{0r}$ is the component of the total rotational vector that is normal to both the velocity vector $\boldsymbol{V}_0$ and $\boldsymbol{A}_{N0}$, and $\boldsymbol{\omega}_{0n}$ is the component of the total rotational vector that is tangent to the velocity vector $\boldsymbol{V}_0$. Thus, $\boldsymbol{\omega}_{0r}$ lies in the plane formed by $\boldsymbol{T} \times \boldsymbol{N}$, and its magnitude can be found as follows:

$$|\boldsymbol{A}_{N0}|\boldsymbol{N} = (\boldsymbol{\omega}_0 \times \boldsymbol{V}_0) = (\boldsymbol{\omega}_{0r} + \boldsymbol{\omega}_{0n}) \times \boldsymbol{V}_0 = \boldsymbol{\omega}_{0r} \times \boldsymbol{V}_0$$

$$= |\boldsymbol{\omega}_{0r}|(\boldsymbol{T} \times \boldsymbol{N}) \times |\boldsymbol{V}_0|\boldsymbol{T} = |\boldsymbol{\omega}_{0r}|\,|\boldsymbol{V}_0|\,\boldsymbol{N}$$

yielding $|\boldsymbol{\omega}_{0r}| = |\boldsymbol{A}_{N0}|/|\boldsymbol{V}_0|$ and $\boldsymbol{\omega}_{0r} = (|\boldsymbol{A}_{N0}|/|\boldsymbol{V}_0|)\boldsymbol{T} \times \boldsymbol{N}$. Solving this expression results in

$$\boldsymbol{\omega}_{0r} = \begin{bmatrix} \dfrac{v_0 a_{z0} - w_0 a_{y0}}{V_0^2} \\ \dfrac{w_0 a_{x0} - u_0 a_{z0}}{V_0^2} \\ \dfrac{u_0 a_{y0} - v_0 a_{x0}}{V_0^2} \end{bmatrix} \tag{9}$$

where $V_0 = V_t = \sqrt{(u_0^2 + v_0^2 + w_0^2)}$. The question now is to determine what is the appropriate choice for $\boldsymbol{\omega}_{0n}$. As $\boldsymbol{\omega}_{0n} \subset \mathfrak{N}[S(\boldsymbol{V}_0)]$, it must be of the form $\boldsymbol{\omega}_{0n} = k[u_0 \;\; v_0 \;\; w_0]^T$, where k is any arbitrary constant. Recalling that we are trying to linearize Eqs. (1–6) about a trim point, it is logical to choose k such that at trim there is no roll rate p_0. This requires that

$$p_0 = \frac{v_0 a_{z0} - w_0 a_{y0}}{V_0^2} + k u_0 = 0 \Rightarrow k = \frac{w_0 a_{y0} - v_0 a_{z0}}{u_0 V_0^2} \Rightarrow \boldsymbol{\omega}_{0n}$$

$$= \left[\frac{w_0 a_{y0} - v_0 a_{z0}}{V_0^2} \quad \frac{(w_0 a_{y0} - v_0 a_{z0})v_0}{u_0 V_0^2} \quad \frac{(w_0 a_{y0} - v_0 a_{z0})w_0}{u_0 V_0^2}\right]^T$$

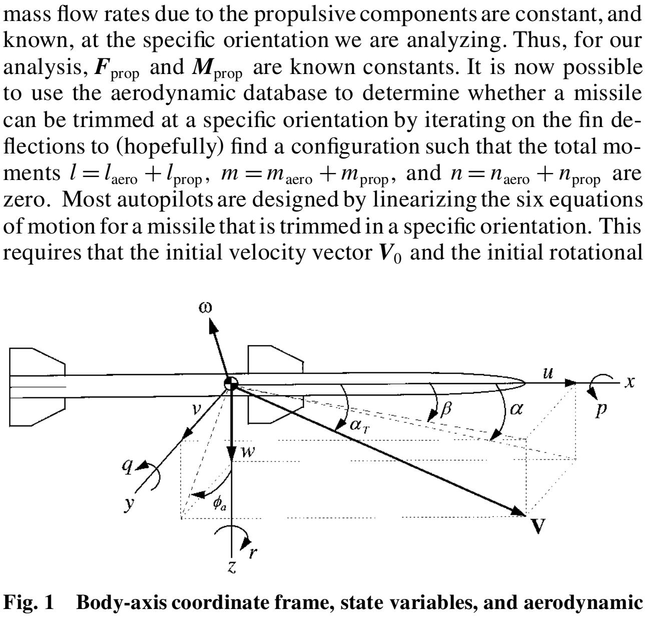

Fig. 1 Body-axis coordinate frame, state variables, and aerodynamic angles.

$$\boldsymbol{\omega}_0 = \begin{bmatrix} p_0 \\ q_0 \\ r_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{u_0 w_0 a_{x0} + v_0 w_0 a_{y0} - \left(u_0^2 + v_0^2\right) a_{z0}}{u_0 V_0^2} \\ \dfrac{-(u_0 v_0 a_{x0} + v_0 w_0 a_{z0}) + \left(u_0^2 + w_0^2\right) a_{y0}}{u_0 V_0^2} \end{bmatrix} \tag{10}$$

Note that it is also possible to zero out other terms in Eq. (10) as opposed to p_0 through proper selection of the vector $\boldsymbol{\omega}_{0n}$. It is now necessary to show that Eqs. (7) and (10) are the proper trim values for $\boldsymbol{V}_0$ and $\boldsymbol{\omega}_0$ and will result in constant values for α and β. To do this, we must show that Eqs. (1–6) must be satisfied at the trim condition as well as $\dot{\alpha}_0$ and $\dot{\beta}_0 = 0$. At trim, Eqs. (1–3) can be solved for $\dot{u}_0$, $\dot{v}_0$, and $\dot{w}_0$ and Eqs. (4–6) can be solved for $\dot{p}_0$, $\dot{q}_0$, and $\dot{r}_0$. This yields

$$\dot{u}_0 = \frac{u_0(u_0 a_{x0} + v_0 a_{y0} + w_0 a_{z0})}{V_0^2} - \frac{\dot{M}_0}{M_0} u_0 \tag{11}$$

$$\dot{v}_0 = \frac{v_0(u_0 a_{x0} + v_0 a_{y0} + w_0 a_{z0})}{V_0^2} - \frac{\dot{M}_0}{M_0} v_0 \tag{12}$$

$$\dot{w}_0 = \frac{w_0(u_0 a_{x0} + v_0 a_{y0} + w_0 a_{z0})}{V_0^2} - \frac{\dot{M}_0}{M_0} w_0 \tag{13}$$

$$\dot{p}_0 = 0 \tag{14}$$

$$\dot{q}_0 = -(\dot{I}_{zz0}/I_{zz0}) q_0 \tag{15}$$

$$\dot{r}_0 = -(\dot{I}_{zz0}/I_{zz0}) r_0 \tag{16}$$

where values for p_0, q_0, and r_0 from Eq. (10) have been used in the equations for $\dot{u}_0$, $\dot{v}_0$, and $\dot{w}_0$. Substituting Eqs. (11–13) into the equations for $\dot{\alpha}_0$ and $\dot{\beta}_0$ results in

$$\dot{\alpha}_0 = \frac{\mathrm{d}\left[\tan^{-1}(w_0/u_0)\right]}{\mathrm{d}t} = \frac{u_0 \dot{w}_0 - w_0 \dot{u}_0}{u_0^2 + w_0^2} = 0$$

$$\dot{\beta}_0 = \frac{\mathrm{d}\left[\tan^{-1}(v_0/u_0)\right]}{\mathrm{d}t} = \frac{u_0 \dot{v}_0 - v_0 \dot{u}_0}{u_0^2 + w_0^2} = 0$$

It has now been shown that Eqs. (7) and (10–16) completely describe a symmetric missile in trim for some orientation (α_t, ϕ_a, Mach, and altitude).

Before linearizing Eqs. (1–6), it is necessary to express the aerodynamic forces $\boldsymbol{F}_{\text{aero}}$ and the aerodynamic moments $\boldsymbol{M}_{\text{aero}}$ in terms of fin deflections and missile orientation. Typically, most autopilots combine the fin deflections into equivalent control surface deflections for pitch (δp), yaw (δy), and roll (δr). Stability derivatives are used as in Ref. 4 to represent $\boldsymbol{F}_{\text{aero}}$ and $\boldsymbol{M}_{\text{aero}}$ in terms of δp, δy, δr, α, and β. For example, $F_{x\text{aero}} = C_{x\alpha}\alpha + C_{x\beta}\beta + C_{x\delta p}\delta p + C_{x\delta y}\delta y + C_{x\delta r}\delta r$, where C_{ij} is the stability derivative of the ith force or moment with respect to j. This representation allows one to express the aerodynamic forces and moments in terms of the variables (u, v, w) due to the presence of α and β. If one chooses the state vector to be $\boldsymbol{x} = [u \;\; v \;\; w \;\; p \;\; q \;\; r]^T$, then Eqs. (1–6) can be written as

$$\dot{x}_1 = -(\dot{M}_0/M_0)x_1 + x_2 x_6 - x_3 x_5 + (1/M_0)\Big[\tan^{-1}(x_3/x_1) C_{x\alpha} + \tan^{-1}(x_2/x_1) C_{x\beta} + C_{x\delta p}\delta p + C_{x\delta y}\delta y + C_{x\delta r}\delta r + F_{x\text{prop}}\Big]$$

$$\dot{x}_2 = -(\dot{M}_0/M_0)x_2 - x_1 x_6 + x_3 x_4 + (1/M_0)\Big[\tan^{-1}(x_3/x_1) C_{y\alpha} + \tan^{-1}(x_2/x_1) C_{y\beta} + C_{y\delta p}\delta p + C_{y\delta y}\delta y + C_{y\delta r}\delta r + F_{y\text{prop}}\Big]$$

$$\dot{x}_3 = -(\dot{M}_0/M_0)x_3 + x_1 x_5 - x_2 x_4 + (1/M_0)\Big[\tan^{-1}(x_3/x_1) C_{z\alpha} + \tan^{-1}(x_2/x_1) C_{z\beta} + C_{z\delta p}\delta p + C_{z\delta y}\delta y + C_{z\delta r}\delta r + F_{z\text{prop}}\Big]$$

$$\dot{x}_4 = -(\dot{I}_{xx0}/I_{xx0})x_4 + (1/I_{xx0})\Big[\tan^{-1}(x_3/x_1) C_{l\alpha} + \tan^{-1}(x_2/x_1) C_{l\beta} + C_{l\delta p}\delta p + C_{l\delta y}\delta y + C_{l\delta r}\delta r + l_{\text{prop}}\Big]$$

$$\dot{x}_5 = -(\dot{I}_{zz0}/I_{zz0})x_5 + I' x_4 x_6 + (1/I_{zz0})\Big[\tan^{-1}(x_3/x_1) C_{m\alpha} + \tan^{-1}(x_2/x_1) C_{m\beta} + C_{m\delta p}\delta p + C_{m\delta y}\delta y + C_{m\delta r}\delta r + m_{\text{prop}}\Big]$$

$$\dot{x}_6 = -(\dot{I}_{zz0}/I_{zz0})x_6 - I' x_4 x_5 + (1/I_{zz0})\Big[\tan^{-1}(x_3/x_1) C_{n\alpha} + \tan^{-1}(x_2/x_1) C_{n\beta} + C_{n\delta p}\delta p + C_{n\delta y}\delta y + C_{n\delta r}\delta r + n_{\text{prop}}\Big]$$

where $I' = (I_{zz0} - I_{xx0})/I_{zz0}$. As these equations are to be linearized about trim for some given missile orientation (α_t, ϕ_a, Mach, and altitude), the initial state vector $\boldsymbol{x}_0 = [u_0 \;\; v_0 \;\; w_0 \;\; p_0 \;\; q_0 \;\; r_0]^T$ is required and can be found from Eqs. (7) and (10). Recall that we are assuming that the forces due to the propulsive elements are known and constant so that the only inputs to the system are the pitch, yaw, and roll fin deflections. The nominal values for these control variables, $\boldsymbol{u}_0 = [p_0 \;\; y_0 \;\; r_0]^T$, are known from the iterations on the aero database. The preceding state equations can be written in compact form as $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ and can be linearized about $(\boldsymbol{x}_0, \boldsymbol{u}_0)$ using a standard Taylor series expansion.[5] Defining $\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{\delta x}$ and $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{\delta u}$ results in the linear perturbation model $\boldsymbol{\delta}\dot{\boldsymbol{x}} = A\boldsymbol{\delta x} + B\boldsymbol{\delta u}$, where

$$A = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}_0, \boldsymbol{u}_0}, \qquad B = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right|_{\boldsymbol{x}_0, \boldsymbol{u}_0}$$

Performing the necessary differentiation yields

$$\boldsymbol{\delta}\dot{\boldsymbol{x}} = \begin{bmatrix} -\dfrac{\dot{M}_0}{M_0} - \left(\dfrac{T_4 C_{x\alpha} + T_2 C_{x\beta}}{M_0}\right) & r_0 + \dfrac{T_1 C_{x\beta}}{M_0} & -q_0 + \dfrac{T_3 C_{x\alpha}}{M_0} & 0 & -w_0 & v_0 \\ -r_0 - \left(\dfrac{T_4 C_{y\alpha} + T_2 C_{y\beta}}{M_0}\right) & -\dfrac{\dot{M}_0}{M_0} + \dfrac{T_1 C_{y\beta}}{M_0} & p_0 + \dfrac{T_3 C_{y\alpha}}{M_0} & w_0 & 0 & -u_0 \\ q_0 - \left(\dfrac{T_4 C_{z\alpha} + T_2 C_{z\beta}}{M_0}\right) & -p_0 + \dfrac{T_1 C_{z\beta}}{M_0} & -\dfrac{\dot{M}_0}{M_0} + \dfrac{T_3 C_{z\alpha}}{M_0} & -v_0 & u_0 & 0 \\ -\left(\dfrac{T_4 C_{l\alpha} + T_2 C_{l\beta}}{I_{xx0}}\right) & \dfrac{T_1 C_{l\beta}}{I_{xx0}} & \dfrac{T_3 C_{l\alpha}}{I_{xx0}} & -\dfrac{\dot{I}_{xx0}}{I_{xx0}} & 0 & 0 \\ -\left(\dfrac{T_4 C_{m\alpha} + T_2 C_{m\beta}}{I_{zz0}}\right) & \dfrac{T_1 C_{m\beta}}{I_{zz0}} & \dfrac{T_3 C_{m\alpha}}{I_{zz0}} & I' r_0 & -\dfrac{\dot{I}_{zz0}}{I_{zz0}} & I' p_0 \\ -\left(\dfrac{T_4 C_{n\alpha} + T_2 C_{n\beta}}{I_{zz0}}\right) & \dfrac{T_1 C_{n\beta}}{I_{zz0}} & \dfrac{T_3 C_{n\alpha}}{I_{zz0}} & -I' q_0 & -I' p_0 & -\dfrac{\dot{I}_{zz0}}{I_{zz0}} \end{bmatrix} \boldsymbol{\delta x} + \begin{bmatrix} \dfrac{C_{x\delta p}}{M_0} & \dfrac{C_{x\delta y}}{M_0} & \dfrac{C_{x\delta r}}{M_0} \\ \dfrac{C_{y\delta p}}{M_0} & \dfrac{C_{y\delta y}}{M_0} & \dfrac{C_{y\delta r}}{M_0} \\ \dfrac{C_{z\delta p}}{M_0} & \dfrac{C_{z\delta y}}{M_0} & \dfrac{C_{z\delta r}}{M_0} \\ \dfrac{C_{l\delta p}}{I_{xx0}} & \dfrac{C_{l\delta y}}{I_{xx0}} & \dfrac{C_{l\delta r}}{I_{xx0}} \\ \dfrac{C_{m\delta p}}{I_{zz0}} & \dfrac{C_{m\delta y}}{I_{zz0}} & \dfrac{C_{m\delta r}}{I_{zz0}} \\ \dfrac{C_{n\delta p}}{I_{zz0}} & \dfrac{C_{n\delta y}}{I_{zz0}} & \dfrac{C_{n\delta r}}{I_{zz0}} \end{bmatrix} \boldsymbol{\delta u} \tag{17}$$

where the terms $T_1 = u_0/(u_0^2 + v_0^2)$, $T_2 = v_0/(u_0^2 + v_0^2)$, $T_3 = u_0/(u_0^2 + w_0^2)$, and $T_4 = w_0/(u_0^2 + w_0^2)$ are due to $\partial\beta/\partial v$, $\partial\beta/\partial u$, $\partial\alpha/\partial w$, and $\partial\alpha/\partial u$, respectively. If the outputs of interest are the accelerations a_y and a_z, and the three body rates p, q, and r, then the nonlinear output equation $\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u})$ is given by

$$\boldsymbol{y} = [(\dot{M}/M)v + \dot{v} + ru - pw \quad (\dot{M}/M)w + \dot{w} - qu + pv \quad p \quad q \quad r]^T$$

The equation $\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u})$ can also be linearized about the trim condition $(\boldsymbol{x}_0, \boldsymbol{u}_0)$ using a Taylor series expansion to obtain $\boldsymbol{\delta y} = C\boldsymbol{\delta x} + D\boldsymbol{\delta u}$, where

$$C = \left.\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}_0,\boldsymbol{u}_0}, \qquad D = \left.\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{u}}\right|_{\boldsymbol{x}_0,\boldsymbol{u}_0}$$

Again, performing the necessary differentiation yields

$$\boldsymbol{\delta y} = \begin{bmatrix} -\left(\dfrac{T_4 C_{y\alpha} + T_2 C_{y\beta}}{M_0}\right) & \dfrac{T_1 C_{y\beta}}{M_0} & \dfrac{T_3 C_{y\alpha}}{M_0} & 0 & 0 & 0 \\ -\left(\dfrac{T_4 C_{z\alpha} + T_2 C_{z\beta}}{M_0}\right) & \dfrac{T_1 C_{z\beta}}{M_0} & \dfrac{T_3 C_{z\alpha}}{M_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{\delta x} + \begin{bmatrix} \dfrac{C_{y\delta p}}{M_0} & \dfrac{C_{y\delta y}}{M_0} & \dfrac{C_{y\delta r}}{M_0} \\ \dfrac{C_{z\delta p}}{M_0} & \dfrac{C_{z\delta y}}{M_0} & \dfrac{C_{z\delta r}}{M_0} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{\delta u} \tag{18}$$

Equations (17) and (18) represent the linear perturbation model of a missile about trim in an orientation specified by (α_t, ϕ_a, Mach, and altitude).

Conclusions

We have presented the equations necessary to obtain a linear time-invariant model of a fully coupled, high-angle-of-attack, six-DOF symmetric missile in trim. The definition of trim used here is that 1) the moments acting on the missile are zero and 2) the time rate of change of both the pitch angle of attack and the sideslip angle is zero. This trim condition results in nonzero initial roll rates that are not unique. A technique for determining the initial roll rates that is consistent with the dynamic equations of a trimmed missile was presented. Both the initial roll rates and the equations for the linear model are valid with and without propulsive forces.

References

[1]Blakelock, J. H., *Automatic Control of Aircraft and Missiles*, 2nd ed., Wiley, New York, 1991, pp. 229–259.

[2]Zhou, K., Doyle, J. C., and Glover, K., *Robust and Optimal Control*, Prentice–Hall, Englewood Cliffs, NJ, 1995, pp. 301–521.

[3]Greenwood, D. T., *Principles of Dynamics*, 2nd ed., Prentice–Hall, Englewood Cliffs, NJ, 1988, pp. 389–393.

[4]Cronvich, L. L., "Aerodynamic Considerations for Autopilot Design," *Tactical Missile Aerodynamics*, edited by M. J. Hemsch and J. N. Nielsen, Vol. 104, Progress in Astronautics and Aeronautics, AIAA, New York, 1986, pp. 17, 18.

[5]Vidyasagar, M., *Nonlinear Systems Analysis*, 2nd ed., Prentice–Hall, Englewood Cliffs, NJ, 1993, pp. 209–219.